## Introduction

The Boolean Satisfiability Problem, also known as Satisfiability or SAT, is the problem of deciding whether there exists an assignment that satisfies a given CNF formula. The main goal is to systematically assign variables values, either TRUE or FALSE, such that the formula evaluates to TRUE. This problem is known to be NP-complete (Cook, 1971), it is the canonical NP-complete problem, and is conjectured not to be solvable in sub-exponential time. (Impagliazzo, Paturi, & Zane, 2001) However, in the vast “ocean” that is satisfiability, there exists some “islands”, called islands of tractability. These islands represent the tractable cases of SAT, i.e. a class of CNF formulas such that the membership of can be decided in polynomial time, or the satisfiability problem for formulas can be decided in polynomial time. (Johannsen, 2020) Many of these cases have been identified and studied in the literature. Some of the most well-known cases are Horn, Co-Horn, 2-CNF, 1-Val, and 0-Val. We will be focusing on some not so well-known classes of tractability.

The class of nested formulas was defined by Knuth (Knuth, 1990) A CNF formula is said to be nested if no two clauses in the set overlap, meaning they can be arranged hierarchically without straddling each other. This hierarchical arrangement allows for a topological sorting of the clauses, which can be exploited to devise efficient algorithms for checking satisfiability. For instance, Knuth devised an algorithm that can decide whether is satisfiable or not in .

Unlike the class of nested formulas, co-nested formulas devised by Kratochvil and Krivanek (Kratochvil & Krivanek, 1993), deal with the graphical properties of a given CNF formula. They define a graph in which there are edges between clauses in order and between the variables and the clauses they occur in. A formula is co-nested if the graph is planar and the clause-to-clause edges lie on the outer face. The authors have devised an algorithm that determines the maximum number of simultaneously satisfiable clauses of a co-nested formula that has a worst case upper bound of .

The main interesting point about these cases are that they pose limitations that lead to interesting properties, such as planarity and non-overlapping clauses. This paper aims to analyze and compare some not so well-known tractable cases, nested and co-nested specifically, and uniformly present them and shed some light on these classes.

## Preliminaries

Before we move on to the tractable cases themselves, some preliminary definitions and notations are in order, we will be starting from the bottom up, defining what Boolean variables and propositional logic, carrying on with the CNF format, explaining the DIMACS format, and finally introducing the DPLL algorithm.

### Boolean Variables and Propositional Logic

Given a Boolean Formula composed of variables and logical connectors, satisfiability in its core asks the question whether there exists an assignment of truth values to the variables such that the formula evaluates to true. To build from the bottom towards the top: we let be a set of variables. Propositional formulas over are defined inductively, where constants and , each variable are formulas. If is a formula, the so is , the negation of . If and are formulas, then so are their conjunction and their disjunction . denotes the set of variables occurring in . represents the size of . Such that:



An assignment for is a finite partial map written as where and . The value of is computed by replacing every by and the simplify according to these rewrite rules:

Figure 1: An exemplary formula in CNF.

Assignment is defined to be total, if , otherwise partial. is said to satisfy , written as , if under this assignment evaluates to true, . is defined to be satisfiable, if for some , and as a tautology, if for all total .

### CNF format

When speaking of satisfiability in general, a formula refers to a formula in Conjunctive Normal Form (CNF). A literal under CNF is a variable or its negation. A clause is a disjunction of literals, for example and a formula is a conjunction of clauses .

The width of a clause is to the number of elements within, therefore the width of clause is . A formula is in if every clause in has width of at most , . With this information in mind, a clause is identified by the set and a CNF formula is identified by the set Throughout this work we will be using to refer to the number of variables in , the number of clauses in , and for the width of .

To further specify the requirement for a CNF formula to be satisfied, if there is a variable for every clause with . In other words, every clause in has at least one variable that evaluates to true.

A unit clause and a pure literal are two special properties that can occur in CNF formulas. Unit clauses are clauses where . Therefore, the literal within the clause needs to evaluate to true for the formula to be satisfiable. A literal is pure in a formula , if the variable occurs only in one polarity. In other words,   does not occur in the formula.

### 2.3 DIMACS Format

Aiming to uniformly represent CNF Formulas, the Center of Discrete Mathematics and Computer Science (DIMACS) at Rutgers University created the DIMACS format for representation of CNF formulas at the 1993 DIMACS challenge. (Johnson & Trick, 1996) The format consisting of a preamble and a body became the norm for the field ever since. (Prestwich, 2009)

The syntax of DIMACS format is as follows. Firstly, a preamble is available containing information regarding the formula. The option to write comments is also available through the usage of the letter ‘c’ at the beginning of a line. Then a line starting with ‘p’ portrays the number of variables and clauses within the formula. Both the variables and the clauses are positive integers. The remainder of the body contains the clauses. Every clause contains a list of non-zero integers, as zero is reserved as a terminator for the clause. The integers representing the variables are numbered from 1 to . Furthermore, the integers must be separated by spaces, tabs, or newlines. In order to represent negative variables a before the variable itself is satisfactory. Finally, the order of the variables within the clauses, and the clauses themselves are not of importance. A clause may even stretch over multiple lines. An example can be found in Figure 2 and 3.

c Pigeonhole principle formula for 3 pigeons and 2 holes

c Generated with `cnfgen` (C) Massimo Lauria <lauria@kth.se>

c https://github.com/MassimoLauria/cnfgen.git

c

p cnf 6 9

1 2 0

3 4 0

5 6 0

-1 -3 0

-1 -5 0

-3 -5 0

-2 -4 0

-2 -6 0

-4 -6 0

Figure 2: An unsatisfiable CNF formula for the pigeonhole principle with 3 pigeons and 2 holes in DIMACS format by cnfgen. The formula consists of 6 variables and 9 clauses.

Figure 3: The same CNF formula found in Figure 2 represented with propositional logic.

### 2.4 Genesis – DPLL

Named after Davis, Putnam, Logemann, and Loveland, the DPLL algorithm is the backbone of many known SAT Solvers, such as but not limited to, zChaff and MiniSat. The algorithm picks branching variables with the aid of backtracking in the case of a conflict. Moreover, DPLL is known to be complete and sound, meaning that it only delivers a solution if and only if the formula is satisfiable. (Prasad, Biere, & Gupta, 2005)

The algorithm works as follows: after simplification steps, an unassigned variable is picked. This variable is assigned either true or false. Here the search tree is branched. If a logical conflict occurs, then the algorithm backtracks and reverts the actions done up until the branching variable. Afterwards the opposite value of the original assignment is done. The algorithm terminates with an assignment if all clauses are satisfied. Otherwise, if both assignments of the initial variable results in a conflict, returns UNSAT. DPLL is a depth-first search where the worst-case runtime is of , which corresponds to the tree size of variables and two choices per node. A pseudo-code for the general DPLL algorithm can be found below.

* General DPLL Algorithm

DPLL(

simplify()

if then return UNSAT

if then return

pick and

DPLL()

if

then return

else return DPLL()

The simplify()function refers to simplification processes that are utilized in order to reduce the number of decisions that need to be taken by the DPLL algorithm and to shorten the formula. Among others, these may include unit propagation (UP), pure literal elimination (PLE), and subsumption.

UP is a branching strategy in which if a variable occurs in a unit clause , then this variable will be picked. If the variable occurs positively, then that clause can be discarded after its assignment, as it is already satisfied. If the variable occurs negatively, then that literal can be discarded after its assignment for that clause, as the satisfiability of the clause is not dependent on it anymore. PLE, similar to UP, is the strategy of picking the variable that is pure in . The aim here is assign the variable in such a way that the clauses it appears in evaluate to true. The pseudo-codes for these strategies can be found below.

* UP Algorithm

UnitProp(

while contains unit clause

return

* PLE Algorithm

PureLit(

while contains pure literal

## Literature Review

Note for Dr Johannsen: There is an extension of nested, I looked and looked could not find it anywhere online. P. Hansen, B. Jaumard, and G. Plateau. An extension of nested satisfiability. Technical Report G-93-27, Les Cahiers du GERAD, 1993.

### Bounded Treewidth

Treewidth is an important graph invariant[[1]](#footnote-1) that measures the “tree-likeness” of a graph. Normally NP-hard problems, such as Hamiltonicity and 3-colorability, are fixed parameter tractable by the treewidth of the input graph. In the sense of satisfiability, given a tree decomposition of a primal graph, whether the input CNF formula is satisfiable or not is decidable utilizing a bottom-up approach on the tree decomposition. (Samer & Szeider, 2009)

“Tree-likeness” or “Horn-likeness” are arbitrary terms that describe the likeness of a CNF formula to a class of formulas. One of the properties these cases have in common is that they associate a non-negative integer with a CNF formula . Such a mapping is called a satisfiability parameter or a parametrization of the satisfiability problem. Bounded treewidth is, for example, one such parameter.

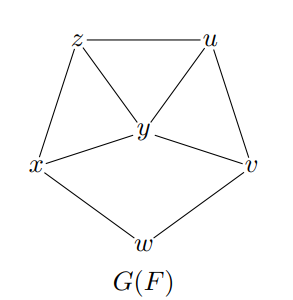


Figure 2: for the CNF, and , ,,

In order to calculate the treewidth of a CNF formula , one must represent the structure of the formula as a graph. One way to do that, among incidence and dual graphs, is the primal graph. In a primal graph the vertices of are the variables of , and two variables are joined by an edge if they occur in the same clause, for some .

Roberston and Seymour studied the tree decomposition and the associated paramter treewidth in their Graph Minors Project work. A tree decomposition of a graph is a tree aside a labeling function . This function associates each tree node a “bag” of of vertices in such that these conditions hold:

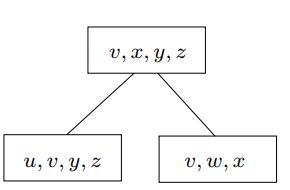


Figure 3: Tree decomposition of the CNF Formula in Figure 2.

1. Every vertex in occurs in some bag ,
2. For every edge there is a bag that contains both variables,
3. If and both contain then each bag contains if lies on the unique path from to .

The treewidth of a graph measures it acyclicity, the lower the treewidth the less cyclic is the graph. In the case that the treewidth equals 1, the is the graph acyclic. Treewidth is calculated with . However, as the computation of treewidth is known to be NP-hard and finding the treewidth for large formulas do not contribute to the efficient decision of satisfiability, graphs with bounded treewidths are more of interest. (S. Arnborg, 1987) Bodlaender has shown that it can be decided in linear time whether a graph has at most treewidth if is a constant. (Bodlaender, 1996)

Given a tree decomposition of a primal graph of a given CNF formula , where is the treewidth of this decomposition. The number of nodes is bounded by r of nodes is bounded by . Each node in the tree decomposition is associated with a table ​ containing rows that represent possible truth assignments to the variables in that node's "bag". These assignments do not falsify any clause in the formula . The size of each table is bounded by , and all tables can be computed in time . This transforms the SAT problem into a constrain satisfaction problem, which can be solved in using Yannakakis’s algorithm. (Yannakakis, 1981) This algorithm, which works in a bottom-up manner on the tree decomposition, iteratively checks for consistent truth assignments between parent and child nodes and prunes inconsistent assignments. If after processing all nodes, there are still valid assignments left in the table associated with the root node, the is the CNF satisfiable. Therefore, the bounded treewidth satisfiability problem can be efficiently solved in . By using an improved algorithm, this time can be shortened to . (Samer & Szeider, 2007)

It should be noted that Samer and Szeider also consider the fixed parameter tractability of incidence graphs and dual graphs in their work as well.

### Horn Derivatives

Horn formulas are a conjunction of Horn clauses, which contain at most one positive literal. Our in-depth analysis of HORNSAT will take place in chapter 4.2. However, a number of derivatives of HORNSAT are available in the literature, such as but not limited to renamable Horn, Extended Horn, and Q Horn. These we will be analyzing in this chapter.

Horn clauses are also of great importance to logic programming, as they are the building blocks of the Prolog language. Horn clauses enable the development of powerful and efficient logic-based systems. Their declarative nature, support for resolution, and predictable behavior make them an ideal foundation for logic programming and reasoning systems.

Given a CNF formula , let be the set of variables and the set of clauses. Then is the result of replacing each literal in literal in with its complement, i.e. is a renaming of . (Lewis, 1978) If some renaming of is Horn, then is renamable Horn.

The class of Renamable Horn is known to be both efficiently decidable and recognizable. A renamable Horn formula is unsatisfiable if and only if one can derive the empty clause by means of unit propagation. (Kleine Büning & Lettmann, 1999) Lewis’s work is based on their theorem: is renamable Horn if and only if is satisfiable.

Let be a set of clauses where each . Then define to be the set of clauses, such that:

Since contains at most two literals per clause, it’s satisfiability can be tested in linear time (as we will be showing in Chapter 4.1), to be exact. A similar result can be achieved with a higher worst case upper bound of by utilizing Aspvall, Plass and Tarjan’s work. (Aspvall, Plass, & Tarjan, 1979)

Extended Horn expressions, introduced by Chandru and Hooker in 1991 (Chandru & Hooker, 205-221), aim to generalize standard Horn formulas. An algorithm, based on a theorem by Chandrasekaran from 1984 (Chandrasekaran, 1984), is provided for finding a model, or satisfying assignment, for a satisfiable extended Horn expression. The algorithm works by first applying unit propagation to simplify the formula, then assigning values to ​ to any remaining unassigned variables, and finally using matrix multiplication to round these values into a final satisfying assignment. However, a significant limitation of this approach is that it can only be reliably applied if the expression is known to be an extended Horn expression. Unfortunately, there is no known polynomial-time method to determine whether a given expression is extended Horn, which limits the practical applicability of the algorithm. As a result, even though unit propagation and matrix multiplication can be done in polynomial time, their detection remain nonpolynomial.

The q-Horn class, originating from Boros, Hammer and Sun, (Boros, Hammer, & Sun, Recognition of q-horn formulae in linear time, 1994) was considered the largest class of polynomial-time solvable expressions that could be succinctly described. This belief was based on a measure called the *satisfiability index*, which applied to an underlying matrix representation of clauses, specifically (0, ±1) matrices. The q-Horn class can be understood using a matrix representation where clauses of a CNF formula are organized into a matrix with specific properties. By multiplying columns by -1, and permuting rows and columns, the matrix can be divided into four quadrants (Truemper, 1998):

* The northeast quadrant contains all zeros.
* The northwest quadrant represents a Horn expression.
* The southeast quadrant represents a 2-SAT expression.
* The southwest quadrant contains no +1s.

An efficient solution for q-Horn expressions can be found by first solving the Horn expression in the northwest quadrant, using the resulting model to eliminate satisfied clauses in the southern quadrants, and then solving the 2-SAT expression in the southeast quadrant. If a model exists, it can be found in linear time. The q-Horn class is defined by the property that the satisfiability index of a CNF expression is no greater than 1 (Boros, Crama, Hammer, & Saks, 45-49). If the satisfiability index exceeds , for any fixed , the problem becomes NP-complete.

#### Backdoors

The current status quo of SAT Solvers is that formulas with hundreds of thousands of variables and millions of clauses can be efficiently solved. However, usually such instances do not belong to one of the tractable cases, creating a discrepancy between theory and practice. One attempt to explain this is that these instances, albeit not a tractable case currently, are close to a tractable case. Meaning that there is a small subset of variables, such that after assigning these variables values, the residual formula transforms into a tractable case. Introduced by Crama et al. (Crama, Ekin, & Hamme, 1997) and the term “backdoor sets” later coined by Williams et al. (Williams, Gomes, & Selman, 2003) consists of several kinds of backdoor sets.

A strong backdoor set for the class is a set of variables, such that every setting of these variables the residual formula is in . A weak backdoor set for is a set of variables, such that some setting of these variables the residual formula is in and is satisfiable. A strong backdoor set of size allows to solve a formula of size in . Dependent on both and , this runtime requires a framework to analyze bounds with fixed-parameter tractability and parametrized complexity, which is handled by Downey and Fellows. (Downey & Fellows, 1992)

is one of the less well-know tractable cases, where each variable has at most two occurrences. The satisfiability of the formulas in this class can be decided in linear time (Kleine Büning & Lettmann, 1999). Finding the backdoor sets with respect to , for both strong and weak, is hard for the class and in both cases becomes fixed-parameter tractable when restricted to for a fixed . (Johannsen, 2020)

Gaspers and Szeider have showed that, parametrized by the size of a smallest strong backdoor set to the base class of nested formulas (which we handle in depth in chapter 4.3), computing the number of satisfying assignments of any CNF formula is fixed-parameter tractable. Therefore, for any , the satisfiability problem can be solved in polynomial time for any for any formula for which there exists a set of at most variables such that every truth assignment to , the reduced formula is nested. Moreover, the degree of the polynomial is independent of . (Gaspers & Szeider, Strong Backdoors to Nested Satisfiability, 2012)

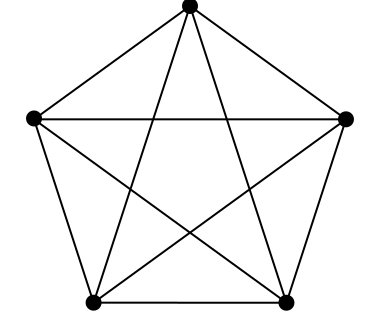
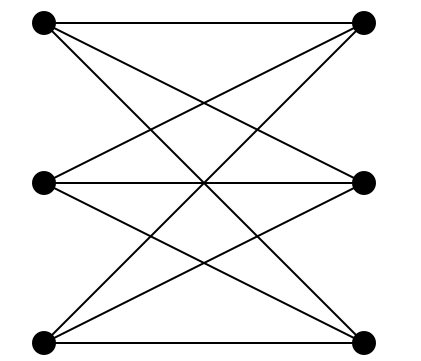


Figure 4: The complete graph (upper) and the utility graph (lower) (Life of Riley)



The algorithm utilizes some refinements from the literature. Firstly, the refactored definition of nestedness of Kratochvil and Krivanek which creates a so called variable linked graph (see chapter 4.4 for the full definition) where the variables are outerplanar and the graph is planar, is used. Secondly, Wagner’s Theorem states that a finite graph is planar if and only if it does not have the complete graph or the utility graph .

Furthermore, using the grid-minor Theorem by Roberston and Seymour, the algorithm either finds that the incidence graph of the formula has bounded treewidth, which can be solved by model checking for monadic second order logic, or finds many vertex-disjoint obstructions (such as the minors or ) in the incidence graph. If obstructions are found then new combinatorial algorithms are used to find a small backdoor set. Combination of both cases creates an approximation of algorithm which produces a strong backdoor set whose size is upper bounded by a function of the optimum. This algorithm is overall upper bounded by , equating to the strong nested backdoor set of size at most . After the set has been found, going through all assignments to this set of variables and using Knuth’s algorithm, it can be decided whether the input formula is satisfiable or not. Along this procedure, like in Knuth’s algorithm, one can also count the number of simultaneously satisfiable clauses of the given formula. Knuth’s algorithm has a worse case upper bound of .

It should be noted that Gaspers and Szeider considers backdoor into other tractable cases, such as Horn, Anti-Horn, 2CNF, and more, in their work “Backdoors to Satisfaction”. (Gaspers & Szeider, Backdoors to Satisfaction, 2012)

## Tractable Cases of SAT and Their Algorithms

In this chapter we will be analyzing the detection and satisfiability of 2-SAT, Horn, Nested, and Co-nested formulas.

#### 2 Sat

2-SAT is the class of formulas in which the width of the formula is equal to at most two, i.e. for every in is true. 2-SAT is known to be one of the trivial tractable cases. (Schaefer, 1978) There exists multiple ways to solve 2-SAT, such as but not limited to, graph-based, random walk, resolution, and unit propagation-based approaches. (Dantsin & Hirsch, 2009) We will be introducing the graph-based algorithm and our version based on DPLL.

In their 1979 article, Aspvall, Plass, and Tarjan devised a linear-time algorithm for this case. Their approach aims to create a directed graph for the formula . contains vertices, representing the variables in and their negations. The edges of are created by utilizing the Implication Law of De Morgan, where the clause is equivalent to   and . The direction of the edges follows the direction of the implication. If there is a unit clause present, then is added as an edge. The algorithm checks the strongly connected components of for cycles that include a variable and its negation at the same time. If and only if that is the case, then is unsatisfiable. For the worst-case upper bound, the algorithm requires where is the number of variables and is the number of edges. (Aspvall, Plass, & Tarjan, 1979)

We have chosen to utilize a different approach than the graph-based algorithm. Our algorithm bases itself on DPLL and uses unit propagation to work through the formula. Starting with an arbitrary variable choice and the assignment of the value, it looks for new unit clauses and stores them in a queue. The disposal of clauses in the case of a positive occurrence and the disposal of literals in the case of a negative occurrence is implemented. These new unit clauses found through the negative polarity literals disposal, will be propagated in the order they were found. These clauses are also seen as forced, as they were created through the assignment of another variable. In the case that there is a logical conflict, we revert the assignments and restore the clauses and assign the opposite value to the variable. If the opposite value for a variable also leads to a conflict, then we terminate and return UNSAT. If no conflict is found, the algorithm carries on assigning autark[[2]](#footnote-3) assignments. With out algorithm we exploit the width of 2-SAT formulas since each assignment of a variable is certain to create a unit clause. Therefore, in the worst case, one full run containing all the variables, , will be satisfactory to determine whether a formula is satisfiable or not. Furthermore, the algorithm either returns a fulfilling assignment or UNSAT. The flowchart in Figure 4 can be seen as a graphical representation of our algorithm. It should be noted that our algorithm carries out the same procedure without the explicit construction of the graph explained above.

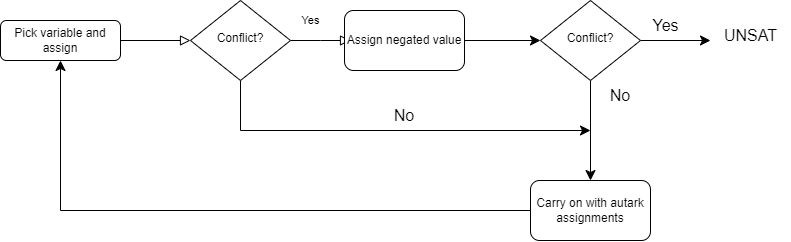


Figure 4: Graphical Flowchart Representation of our 2 SAT algorithm

It should be noted that during parsing if we find a unit clause, we store that value in a queue. The chosen branching strategy is incremental; however, it only comes into effect if there are no other elements remaining in the queue. Therefore, our algorithm assigns more importance to original unit clauses and the unit clauses created by extension.

#### Horn Formulas

One of the classes of the tractable cases of SAT is Horn Satisfiability, also called HORNSAT. The name stems from Alfred Horn, who in a 1951 article pointed out their importance. Unlike general SAT, HORNSAT defines restrictions regarding the polarity of variables in clauses and how the general structure of the formula is built up. A Horn formula is a conjunction of Horn clauses, which contain at most one positive literal. (Downing & Gallier, 1984)

Figure 5: An exemplary Horn formula

The satisfiability of positive Horn formulas can be tested as follows. If contains positive unit clauses, then apply unit clause elimination until no unit clauses are left. Therefore, assigning all positive unit literals the value true and all negative unit literals the value false. If the resulting formula does contain the empty clause, then is unsatisfiable. Otherwise, the clauses within all must contain at least two literals and at least one of the clauses has to be negative. Consequently, is satisfiable by assigning false to all the remaining literals. By extension, is satisfiable as well. One can trivially see that this algorithm is polynomial, to be exact, where represents the number of variables and the number of clauses. (Dantsin & Hirsch, 2009)

We have tweaked the algorithm for the sake of simplicity and readability. In our version the algorithm works as follows. Starting with the empty assignment , iteratively assign all positive unit clauses to true, and then assign the remaining variables to false. If this assignment satisfies then return . Otherwise return UNSAT. Such that, the pseudo-code for our algorithm is as follows:

* HORN ALGORITHM

while positive unit clause in

if

then return

else return UNSAT

Our algorithm, alike its counterpart explained earlier, has a worse case upper bound of . However, instead of doing complete unit propagation, we have chosen to only propagate the positive unit literals. This, combined other optimization approaches, such as marking unit clauses during parsing, can lead to improved performance.

One of the important properties of this algorithm and Horn class is that, if the formula and is satisfiable, then it has a unique minimal model[[3]](#footnote-4) with respect to 1. Meaning that this algorithm finds the satisfying assignment that only assigns the variables that are absolutely required to be true for the formula to be satisfied. (Biere, Heule, Van Maaren, & Walsh, 2021, p. 28)

### Nested Satisfiability

Knuth in his 1990 paper titled “Nested Satisfiability” explores a special case of the satisfiability problem where if a formula has a specific hierarchical structure, that formula gets transformed into Dynamic 2 SAT and therefore is solvable in linear time. He acknowledges the work of Lichtenstein where it was proved, that the joint satisfiability problem of two sets of nested clauses is NP-complete. (Lichtenstein, 1982). However, his exploration shows that the mirrored question for nested clauses is efficiently decidable. (Knuth, 1990).

In order to achieve this, a linear order through is defined, where is a finite alphabet of Boolean variables. Here literals over are elements of the form or where . Literals belonging to are called positive and negative otherwise.

To refine this linear order, linear preordering is introduced where all the literals are arranged in a “natural way”, disregarding the signs. As an example, if , then . If and are literals, then the relational operation can only be true, if or is true and the relation is false.

A clause over is defined as a set of literals on distinct variables, such that the clause can be written in increasing order . The set of clauses over is satisfiable, if there exists a clause over that has a nonempty intersection with every clause in . For example: the clauses in :

are satisfiable by the clause , which has a nonempty intersection with each clause.

A clause straddles , if there are literals and in and in such that Two clauses overlap if they straddle each other. For example, for and :

straddles , since: there exists and and , such that .

straddles , since: there exists and and , such that .

Therefore, and overlap. Clauses that have only 2 elements each, in other words are E2-CNF, such as and can be overlapping. A set of clauses in which no two overlap is defined to be *nested*.

Regarding the structure of these clauses and the way they are represented further refinements are made. A clause over an ordered alphabet has a least literal and a greatest literal . (Knuth, 1990) Any other variable that lies strictly between these literals is defined to be interior to that clause. Since the definition of nestedness sets out the condition for the clauses to not be overlapping, a literal can occur as an interior literal on at most one of the clauses in the set of clauses, otherwise those clauses would be overlapping. This property forces the number of total elements among nested clauses on variables to be . In detail: 1 least and 1 greatest literal per clause and then all variables occur once as an interior literal, therefore . Moreover, this property further limits a possible CNF formula to be a nested formula. In the case that the clause widths are fixed to , i.e. , no formula is nested unless the clause density measure holds.

Furthermore, a transitive relational operation is introduced where represents that straddles but does not straddle . The transitive property of this relation makes it possible to topologically sort any set of nested clauses into a linear arrangement where each clause appears after the clause it straddles. With such an arrangement and the elements presented in order, Knuth defines an algorithm that decides in steps where is the number of clauses and is the number of variables.

#### Nested SAT Algorithm

The presented algorithm determines whether a given nested formula is satisfiable or not using a table to keep track of the satisfiability conditions across intervals of literals. It partitions the literals into intervals, which represent clauses, and updates the sat table to reflect whether the processed clauses can be satisfied based on the literal’s polarity.

The alphabet is assumed to represent the positive integers up to , where each variable is deemed as equivalent to its negation, . The clauses are then specified in two arrays: one where all the literals are present called , and one called signifying the starting positions of a given clause within such that:

and

where the literals of clause are between

in increasing order as increases. The clauses are ordered in such a way that the clause does not straddle clause when . It is assumed that the clauses contain at least two literals.

As the algorithm progresses through each clause, it conceptually divides the set of all encountered clauses so far into intervals . This ensures that all literals of any previously processed clause are in one of these defined intervals. The current intervals are kept in an array, ,where for .

Essentially the heart of the algorithm, the array is interpreted as follows: If is an interval of the current partition, then [] will either be 0 or 1 for each pair . It is 1 if and only if the clauses already processed, belonging to interval , are satisfiable by clauses in which the least and greatest literals are respectively and , where:

As an example, assume we have only seen the clause . Then the sat table looks as follows

The main task of the algorithm is to maintain the array as a new clause is examined. The variables of , belong to the current partition of the variables. Assume . The algorithm runs a variable through the values and keeps the information to update where the interior variables within are eliminated within the partition. We further let be the literals element of that are strictly less than and let be the clauses that are preceding whose literals are limited by . Then an auxiliary table named , with the help of a function, updates the values as follows:

Concretely, means that there is a clause with and that has a nonempty intersection with each clause of the given set of clauses.

For example, suppose and . Furthermore, suppose that clauses have led to the following values:

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  |  |  |  |
|  |  | 0 | 0 | 0 |
|  |  | 1 | 1 | 0 |
|  |  | 1 | 1 | 0 |
|  |  | 1 | 0 | 1 |

Table 1: table for the example

Then the table looks as follows:

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  |  |  |  |  |  |
|  |  | 1 | 0 | 2 | 0 |
|  |  | 0 | 2 | 0 | 0 |
|  |  | 0 | 1 | 2 | 0 |
|  |  | 1 | 1 | 1 | 1 |

Table 3: newsat table for the example

Table 3: Updated sat table after newsat calculation.

In this case, a 1 is the current , the values for will be updated with those in Table 3 and will be assigned 4 as interior variables will be discarded.

If the clause were instead, the computation of the table would have been the same, however the values of would all be 0 and the clauses would be unsatisfiable since is equal to 1 and not 2.

The algorithm itself starts with the setup of the array and initializes all values to 1. Then the array is initialized as explained and partitions the formula into its clauses. For each clause, the literals that are in the partition are determined and the table is computed. Depending on whether the new clauses allow a satisfiable combination of the literals in the current partition, table is used to update the array. Here only the least and greatest literals are considered as interior literals are removed from the partition. After iterating through all clauses, the algorithm checks the final status of the table and determines whether the entire set of clauses is satisfiable and outputs SAT or UNSAT accordingly.

The runtime of the algorithm, as explained above, is since each value of is either the first or the last in the current clause or will be permanently removed from the partition.

Although the algorithm provides a viable solution to the problem, its assumptions and the lack of variable assignments in the output present limitations. It is possible to deduce the variable assignments through an analysis of the table after the algorithm delivers a result, however, this would involve a similar approach to DPLL as one must check four values for each clause. During this process logical conflicts can occur, which requires backtracking to be implemented alongside the usual assignment structures. Therefore, either external help from other solvers or an effort to implement a DPLL-like algorithm is required.

Furthermore, as it is also noted by the authors that whether the formula is nested in some ordering of its variables and clauses is not considered. Firstly, that process requires that the variables in each clause to be ordered by the hierarchy. Depending on what kind of ordering algorithm is used, this will require some overhead. Moreover, the clauses themselves are assumed to be ordered in such a way that the clause does not straddle if , again leading to an overhead. Finally, the nestedness condition, i.e. no overlapping clauses, requires that each clause is checked against all the other clauses. Such a comparison requires pairwise checking, which has a time complexity of .

In summary, this algorithm answers the question, whether a given nested formula is satisfiable or not with a worse case upper bound of Although it must be noted that the assumptions made and the prerequisite that a formula to be nested does require further checks and structures to be built.

### Co-nested Formulas

In Kratochvil and Krivanek’s work titled “Satisfiability of co-nested formulas” they introduce a graph-based approach to define Knuth’s nestedness term and define new types of nestedness, specifically co-nestedness, double nestedness and double co-nestedness. Their work assumes the usual prerequisites for CNF formulas adapted for SAT, such that is a formula with a set of clauses over a set of variables . (Kratochvil & Krivanek, 1993)

The main extension for Knuth’s work begins with the definition of a so-called clause linked graph of where . The redefinition of Knuth’s nested formula is as follows: is nested if the *variables* can be ordered in a way where for the graph allows a noncrossing drawing ( is planar[[4]](#footnote-5)) in the plane so that the circle of variables bounds the outer face. The definition of co-nestedness is made in a similar way where the clause linked graph allows a noncrossing drawing in the plane such that the clauses bounds the outer face. A recursive algorithm is defined which computes the maximum number of satisfiable clauses in a given co-nested formula. The runtime of this algorithm is set to be linear in the number of clauses , added with the number of variables . This algorithm will be analyzed in the next chapter.

Furthermore, the notion of double co-nestedness is introduced. is double co-nested if is planar, i.e. the double co-nested formula can be splitted into two co-nested formulas and such that for . It is acknowledged that the satisfiability for double co-nested (double nested formulas is NP-complete, since Lichtenstein proved that even if every clause contains at most three variables, every variable occurs in exactly three clauses and the variables only occur once negatively and twice positively (Lichtenstein, 1982).

#### Co-Nested Maximum Satisfiability Algorithm

The algorithm determining the maximum number of clauses that can be simultaneously satisfied requires some definitions to be made before analyzing the algorithm itself. Given a co-nested formula where its clauses admit a planar drawing of such that the circle bounds the outer face. For each variable , the degree of is defined as and the indices of the clauses containing or . For example, if the variable 1 occurs 3 times in different clauses, the its degree is and refers to the first occurrence (conversely refers to the last occurrence of . In both cases a clause index is returned.

The maximum number of clauses of that can be simultaneously satisfied is denoted by . With these information 2 Lemmas are defined:

* **Lemma 1**: Let . Then
* **Lemma 2**: Let and and . Then . Then .

Lemma 1 essentially undertakes pure literal elimination with a single occurrence, since if a variable occurs only once, then there exists an assignment depending on its polarity that satisfies that clause as a whole. Lemma 2 considers duplicate clauses where the variables occurring in them do not occur in any other clause. Therefore, the two clauses in which variables and occur, can be satisfied. In both cases since the clauses are satisfied, is incremented by the respective values and the clauses are removed from . After the utilization of Lemma 1 and 2 it is assumed that all variables have a degree greater than one and no two variables which occur in exactly two clauses are present in the same pair of clauses.

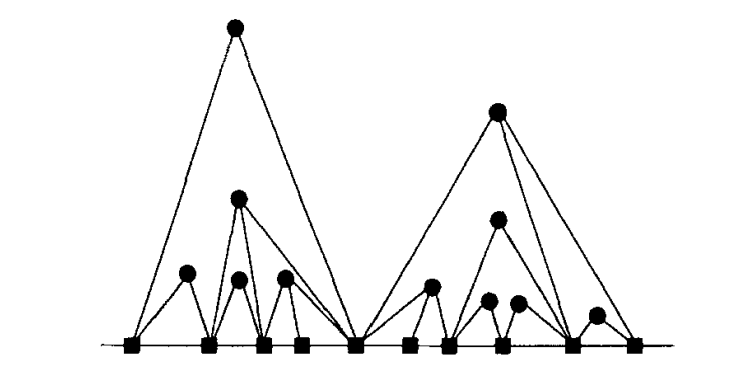


Figure 6: The canonical drawing of a co-nested formula (Kratochvil & Krivanek, 1993, p. 399)

To ease the understanding, a canonical drawing of co-nested formulas is given. Here is assumed to have the clause-to-clause edges in a line and all edges are straight. Clauses are represented by squares and the variables by circles.

Moreover, two partial orders on the set (the set of variables of ) are defined. Firstly, if . Meaning that the clause index of the clause where occurs last is less than or greater than the clause index of first occurrence, so stops occurring either at the same time or before first occurs. Secondly, if and . Meaning that starts occurring at same time or earlier than and stops occurring either at the same time or before stops occurring. Furthermore, if is a direct predecessor[[5]](#footnote-6) of in , then this is denoted by (respectively if direct predecessor in ). One could understand that the partial order is “horizontal” one and a “vertical” one. The partial order partitions the variable set into levels: set to be maximal elements[[6]](#footnote-7) and then recursively for , is the set of maximal elements in Lastly, if and belong to the same level , holds, and there is no variable and index such that , then is written. Then   is defined by if and only if or is an equivalence relation on .

* **Lemma 3**: Each vertical level is linearly ordered by the “horizontal” order and is splitted into (possibly multiple) equivalence classes of .

The graph of a co-nested formula creates a mountain-like structure. Therefore, each formula can be seen as a range of hills, where each hill is a subformula for some variable . Each hill then divides into a range of hillsides, which corresponds to the triangles . Each hillside reveals a new range of higher level whose peaks form an equivalence class .

The algorithm utilizes this idea about the structure of co-nested formulas and computes the for some with the information from for . However, further technical definitions are required.

* is the set of variables that lie inside the triangle , *.*
* spans all variables within the triangle ,
* = : the subformula containing the clauses on the variables from the triangle , (including ), restricted to
* is the subformula containing the clauses on variables from restricted to .
* For such that :
* And for .

Further two recursive functions are defined. The function is an integer function that assigns each triple the maximum number of satisfied clauses in taken over all true/false assignments of the variables of such that and , if no such assignment exists is set.

Defined analogously with respect to , the function returns the maximum number of satisfied clauses in taken over all true/false assignments of the variables of such that and .

The function is called with:

where is the variable with and the largest , and is the variable with and smallest . To express it differently, is the minimal element in and the maximal element in .

As a note it should be recognized that after this point if is a Boolean expression, then (or 0 respectively) if is true.

The formal definition g called for a range of hills, and , is as follows:

The formal definition of splits itself into two functions. First one being, for , is the maximum number of clauses satisfied in where the maximum is taken over all assignments to variables such that , . With respect to , will be used, which calls another function

For , is the function that assigns the singular variables values and checks whether the clauses are fulfilled. It looks as follows:

In the case that is empty, which can be empty as the variables of the clauses that line on the triangle are not included, . If is not empty, then denoting (and respectively) are the -minimal (-maximal respectively) variable in . This function considers four cases, which are illustrated in Figure 7. With , is equal to:

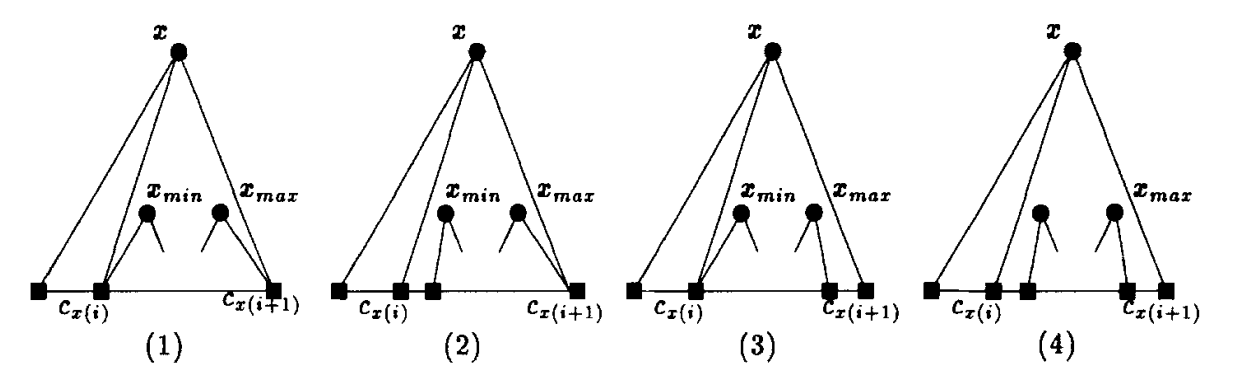


Figure 7: Four cases for

For is , which equates to . Finally, is obtained.

Because of the recursive composure of and ( calls , calls and with a maximal or minimal element, and so on), the hills and hillsides of the co-nested formula will be traversed for a constant amount. Since taking the maximum over different values for , and their derivates only cause this amount to be multiplied by a constant number, which is ignored in the notation, the worst case upper bound for this algorithm is the number of variables added with the number of clauses, i.e.

However, the single assumption of the algorithm is that is a co-nested formula, which means that the clauses of can be ordered in such a way that the circle , bound the outer face and that allows a noncrossing drawing in the plane, i.e. is planar, requires significant computation as the number of clauses grows. Therefore, the question “Given a CNF formula , is there an ordering of its clauses such that the clause linked graph is planar with its clauses bounding the outer face?” poses itself. Similarly, the same question for nested formulas with the variable linked graph should be considered. These questions are equivalent to the problems

PLANAR BI-HAMILTONION ORDERING:

INSTANCE: A planar bipartite graph with color classes .

QUESTION: Can the vertices of the color class A be ordered so that the graph is planar?

OUTERPLANAR BI-HAMILTONION ORDERING:

INSTANCE: A planar bipartite graph with color classes .

QUESTION: Can the vertices of the color class A be ordered so that the graph is outerplanar with all the vertices of lying on the outer face?

The outerplanar bi-Hamiltonian ordering can be decided in linear time by adding a new vertex to a given bipartite graph and connect it to all vertices of In the case nested formulas, represent the variable-to-variable edges, whereas in co-nested formulas corresponds to the clause-clause edges. Then the new graph is planar if and only if the vertices of admit an ordering such that is planar with all the vertices of bounding the outer face. Therefore, in such a case the depth-first-search-based linear planarity checking algorithm of Hopcroft and Tarjan can be utilized to check whether the graph (, respectively) is planar with the clause chain (variable chain respectively) bounding the outer face (Hopcroft & R.E., 1971). This algorithm has a worst case upper bound of where is the number of variables and the number of edges. Therefore, co-nestedness (nestedness respectively) of an ordering of CNF formula can be decided in linear time.

However, it should be noted that for both cases the current ordering of the clauses of must allow these conditions to be met. That is not always the case. If the original ordering does not admit such an embedding, then that only determines the current ordering to be not co-nested (nested respectively). In order to fully check whether a formula can admit such a drawing, one must check all orderings of the clauses, which involves permutating the clauses until one is found. Even though, this operation for very few clauses does not pose a challenge for modern day computers, even as early as 8 clauses the amount of possible ordering that need to be checked exceed 40 thousand in the worst case. This permutation alone has a worst case upper bound of where is the number of clauses. The remaining per run needs to be added on top as it will be checked per ordering, therefore the total worst case upper bound to determine whether a CNF formula is co-nested (nested, respectively) is .

Similarly to the nested algorithm, this algorithm only delivers the number of simultaneously satisfiable clauses, but not their singular assignments. One can utilize memorization and keep track of the assignments, which is an improvement to the nested algorithm. However, that does imply the need to build further structures.

In summary, this algorithm answers the question, whether a given co-nested (nested, respectively) formula in CNF is satisfiable or not with a worse case upper bound of Although it must be noted that the assumptions made and the prerequisite raised regarding and to be planar and the clause to clause chain (variable to variable chain, respectively) to be bounding the outer face does significantly increase the difficulty of recognizing such formulas with a worst case upper bound of .

## Conclusion

As a conclusion, we have analyzed 2-SAT, Horn, Nested, and Co-nested formulas and their classes in depth.

For the case of 2-SAT, their recognition involves going through each clause in a CNF formula and determining whether all clauses have at most 2 variables. The satisfiability of CNF formulas in this class can be determined in by using the graph-based algorithm, or in by utilizing a DPLL like algorithm that uses unit propagation and autark assignments.

For the case of Horn formulas, each clause must hold at most one positive literal. This condition needs to checked for all clauses. If a given CNF formula is Horn, then the satisfiability of can be decided in , resulting in a minimum model if any can be found.

In the case of nested formulas, there should exists no two clauses that overlap each other, this procedure can be done in . However, one might want to utilize Kratochvil and Krivanek’s definition of being planar and the variable chain to be outerplanar. That would require in the worse to be checking all permutated orders of the variables and a depth-first based search for the planarity check, in total amounting to a worse case upper bound of . If the formula is nested, then Knuth’s algorithm can be used to determine the satisfiability in . This algorithm returns the number of simultaneously satisfiable clauses, therefore if that number equals , the formula is satisfied. However, the limitation that the maximum number of elements cannot exceed does decrease the chance of a formula to be nested in practical uses. Furthermore, as the algorithm is designed to only return the number of satisfied clauses, in order to gain information about singular assignments, one must build further structures.

Finally, for the case of co-nested formulas, must be planar and the clause chain needs to be outerplanar for one ordering of its clauses. This, like nested formulas, implies that in the worst case all permutation of the ordering of the clause must be considered, and then a planarity check must be done. This equates to a worse case upper bound of . If the formula is co-nested, then Kratochvil and Krivanek’s can be utilized to determine the maximum number of simultaneously satisfiable clauses in . However, the planarity constraint and the rigorous check that comes with it requires significant computing power. Moreover, like in nested formulas, only a number of satisfied clauses is returned.

We hope that we have shed some light to the cases of nested and co-nested formulas and hope that further research into the graphical properties of these cases can illuminate more hidden factors in the shadows.

Open Questions for future researchers:

* Does nestedness imply co-nestedness and vice-versa?
* Other than checking for planarity for each permutation of and , is there a better way to recognize that a CNF formula is nested or co-nested?

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1. A graph invariant is a property of graphs that only depends on the abstract structure, and not on graph representation, labeling, or drawing (Lovász, 2012). [↑](#footnote-ref-1)
2. An autark assignment is a partial assignment that satisfies a clause with a variable fixed by it. (Kimura & Makino, 2018) [↑](#footnote-ref-3)
3. A minimal model is the intersection of all satisfying assignments of the formula. (Biere, Heule, Van Maaren, & Walsh, 2021, p. 28) [↑](#footnote-ref-4)
4. A graph is planar if it can be drawn in a plane without graph edges crossing (West, 2001) [↑](#footnote-ref-5)
5. A direct or immediate predecessor of an element is an element such that is less than and there is no element such that . [↑](#footnote-ref-6)
6. A maximal element refers to the element in an ordered (or partially ordered) set where there exists no other element such that . A minimal element conversely refers to the element where for all other elements , holds. (Davey & Priestley, 2002) [↑](#footnote-ref-7)